Proof-reading

Check that whether the following method is correct or not.

Given $f(x)=\sqrt{\frac{x}{1+x^2}}$, $x \geq 0$, find its

(a) optimum point(s) ;

(b) point of inflection.

(a) Let $f(x)=\sqrt{g(x)}$, $g(x)=\frac{x}{1+x^2}$, $x \geq 0$

$$g'(x)=\frac{(1+x^2)(1)-x(2x)}{(1+x^2)^2}=\frac{1-x^2}{(1+x^2)^2}=0 \Rightarrow x=1 \quad (x=-1 \text{ is rejected})$$

When $0<x<1$, $g'(x)<0$ and when $x>1$, $g'(x)>0$.

By the First Derivative Test, $g(x)$ is a local max. when $x=1$.

Local Max. of $g(x)=\frac{1}{1+1^2}=\frac{1}{2}$.

Since $\lim_{x\to+\infty} f(x)=\lim_{x\to+\infty}\sqrt{\frac{x}{1+x^2}}=\lim_{x\to+\infty}\sqrt{\frac{1}{\frac{1}{x}+x}}=0$ and $f(0)=0$.

Hence, the absolute maximum of $f(x)=\sqrt{g(x)}=\sqrt{\frac{1}{2}}\approx \mathbf{0.707107}$ when $x=1$.

$\therefore$ $f(0)=0$ is the absolute minimum.

(b) $g'(x)=\frac{1-x^2}{(1+x^2)^2}=\frac{2-(1+x^2)}{(1+x^2)^2}=\frac{2}{(1+x^2)^2}-\frac{1}{1+x^2}$

$$g''(x)=-\frac{8x}{(1+x^2)^3}+\frac{2x}{(1+x^2)^2}=\frac{-8x+2x(1+x^2)}{(1+x^2)^3}=\frac{2x(x^2-3)}{(1+x^2)^3}=0$$

$\therefore x=0, \pm\sqrt{3}$

Since $x \geq 0$, and obviously when $x=0$, $g(x)$ is not a point of inflection, $\therefore x=\sqrt{3}$.

Also $g''(x)$ changes sign when x goes through $\sqrt{3}$.

Therefore when $g(\sqrt{3})$ is a point of inflection.

$$f(\sqrt{3})=\sqrt{g(\sqrt{3})}=\sqrt{\frac{\sqrt{3}}{1+(\sqrt{3})^2}}=\frac{\sqrt[4]{3}}{2}$$

$\therefore$ Point of inflection is $\left(\boldsymbol{\sqrt{3}}, \frac{\boldsymbol{\sqrt[4]{3}}}{\mathbf{2}}\right)$.